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Multiplicity of solutions for a class of nonsymmetric eigenvalue hemivariational inequalities[1]

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Abstract. The aim of this paper is to establish the influence of a non-symmetric perturbation for a symmetric hemivariational eigenvalue inequality with constraints. The original problem was studied by Motreanu and Panagiotopoulos who deduced the existence of infinitely many solutions for the symmetric case. In this paper it is shown that the number of solutions of the perturbed problem becomes larger and larger if the perturbation tends to zero with respect to a natural topology. Results of this type in the case of semilinear equations have been obtained in [1] Ambrosetti, A. (1974), *A perturbation theorem for superlinear boundary value problems*, Math. Res. Center, Univ. Wisconsin-Madison, Tech. Sum. Report 1446; and [2] Bahri, A. and Berestycki, H. (1981), A perturbation method in critical point theory and applications, *Trans. Am. Math. Soc.* 267, 1–32; for perturbations depending only on the argument.

Key words: Critical point theory; Essential value; Hemivariational eigenvalue problem; Perturbation from symmetry

1. Introduction and the main result

The study of variational inequality problems began around 1965 with the pioneering works of G. Fichera, J.-L. Lions and G. Stampacchia (see [7, 8]). The connection of the theory of variational inequalities with the notion of subdifferentiability of convex analysis was achieved by J.J. Moreau (see [9]) who introduced the notion of convex superpotential which permitted the formulation and the solving of a wide ranging class of complicated problems in mechanics and engineering which could not until then be treated correctly by the methods of classical bilateral mechanics. All the inequality problems studied to the middle of the ninth decade were related to convex energy functions and therefore were firmly linked with the notion of monotonicity; for instance, only monotone, possibly multivalued boundary conditions and stress–strain laws could be studied. In order to overcome this limitation, P.D. Panagiotopoulos introduced in [14, 15] the notion of nonconvex superpotential by using the generalized gradient of F.H. Clarke. Due to the lack of convexity new types of variational expressions were obtained. These are the so-called *hemivariational inequalities* and they are no longer connected with

[1] This paper is dedicated to the memory of Professor P.D. Panagiotopoulos.

monotonicity. Generally speaking, mechanical problems involving non-monotone, possibly multivalued stress–strain laws or boundary conditions derived by nonconvex superpotentials lead to hemivariational inequalities. Moreover, while in the convex case the static variational inequalities generally give rise to minimization problems for the potential or the complementary energy, in the nonconvex case the problem of substationarity of the potential or the complementary energy at an equilibrium position emerges.

For a comprehensive treatment of the hemivariational inequality problems we refer to the monographs Panagiotopoulos [16, 17], Motreanu-Panagiotopoulos [11] and Naniewicz-Panagiotopoulos [12].

Throughout this paper V will denote a real Hilbert space which is densely and compactly imbedded in $L^p(\Omega;\mathbb{R}^N)$, for some $1<p<\infty$ and $N\geqslant 1$, where Ω is a bounded domain in $\mathbb{R}^m$, $m\geqslant 1$. Let $p'=p/(p-1)$ be the conjugated exponent of p. Denote by $\|\cdot\|$ the norm on V and by $(\cdot\,,\cdot)$ the corresponding inner product. Let $a:V\times V\to\mathbb{R}$ be a continuous, symmetric and bilinear form, not necessarily coercive. We denote by $A:V\to V$ the self-adjoint bounded linear operator corresponding to a, i.e.

$$(Au, v) = a(u, v) \quad \text{for all } u,\ v\in V\,.$$

Denote by $|\cdot|$ the Euclidian norm in $\mathbb{R}^N$, while the duality pairing between V^* and V (resp., between $(\mathbb{R}^N)^*$ and $\mathbb{R}^N$) will be denoted by $\langle\cdot\,,\cdot\rangle_V$ (resp., $\langle\cdot\,,\cdot\rangle$). For $r>0$, set S_r the sphere of radius r in V centered at the origin, i.e.

$$S_r=\{u\in V;\ \|u\|=r\}\,.$$

Let $j:\Omega\times\mathbb{R}^N\to\mathbb{R}$ be a Carathéodory function which is locally Lipschitz with respect to the second variable and such that $j(\cdot,0)\in L^1(\Omega)$. Thus, we can define the directional derivative in the sense of Clarke (see [4])

$$j^0(x,\xi;\eta)=\limsup_{(h,\lambda)\to(0,0+)}\frac{j(x,\xi+h+\lambda\eta)-j(x,\xi+h)}{\lambda},\ \text{for } \xi,\eta\in\mathbb{R}^N\,,$$

and the Clarke generalized gradient

$$\partial_y j(x,y)=\{w\in(\mathbb{R}^N)^*;\ \langle w,\eta\rangle\leqslant j^0(x,y;\eta)\ \ \forall\eta\in\mathbb{R}^N\},\ \ (x,y)\in\Omega\times\mathbb{R}^N\,.$$

Assume further that this functional satisfies the following conditions:

($\mathbf{A}_1$) $j(x,y)=j(x,-y)$, for a.e. $x\in\Omega$ and every $y\in\mathbb{R}^N$.

($\mathbf{A}_2$) there exist $a_1\in L^{p/(p-1)}(\Omega)$ and $b\in\mathbb{R}_+$ such that
$|w|\leqslant a_1(x)+b|y|^{p-1}$,
for a.e. $(x,y)\in\Omega\times\mathbb{R}^N$ and all $w\in\partial_y j(x,y)$.

Consider $\Lambda:V\to V^*$ the duality isomorphism

$$\langle\Lambda u, v\rangle_V=(u,v)\,,\quad\text{for all } u,v\in V.$$

Suppose also that the following assumption holds

($\mathbf{A}_3$) For every sequence $(u_n) \subset V$ with $\|u_n\| = r$, for every number $\alpha \in [-r^2\|A\|, r^2\|A\|]$,

and for every measurable map $w : \Omega \to (\mathbb{R}^N)^*$ such that

$$u_n \to u \text{ strongly in } L^p(\Omega; \mathbb{R}^N) \text{ for some } u \in V$$
$$w(x) \in \partial_y j(x, u(x)) \quad \text{for a.e. } x \in \Omega \quad \text{and} \quad a(u_n, u_n) \to \alpha$$

then

$$\inf_{\|\tau\|=1} \{a(\tau, \tau)\} - r^{-2}\left(\alpha + \int_\Omega \{w(x), u(x)\rangle \, dx\right) > 0 .$$

Consider the following eigenvalue problem

$$(\mathbf{P}_1)\begin{cases}(u, \lambda) \in V \times \mathbb{R} \\ a(u, v) + \int_\Omega j^0(x, u(x); v(x)) \, dx \geqslant \lambda(u, v), \quad \text{for all } v \in V \\ \|u\| = r.\end{cases}$$

Under assumptions ($\mathbf{A}_1$)–($\mathbf{A}_3$), Motreanu and Panagiotopoulos proved in [10], Theorem 2 that this problem admits infinitely many distinct pairs of solutions $(\pm u_n, \lambda_n)_{n \geqslant 1} \subset S_r \times \mathbb{R}$ with

$$\lambda_n = r^{-2}\left(a(u_n, u_n) + \int_\Omega \langle w_n(x), u_n(x)\rangle \, dx\right), \quad n \geqslant 1 ,$$

where $w_n : \Omega \to (\mathbb{R}^N)^*$ denotes a mapping such that $\langle w_n, u_n\rangle \in L^1(\Omega, \mathbb{R})$ and

$$w_n(x) \in \partial_y j(x, u_n(x)) \quad \text{for a.e. } x \in \Omega .$$

Remark that they assumed $a_1 = \text{const.}$ in ($\mathbf{A}_2$) such that their statement is done under a slightly less general hypothesis. We observe that in order to show that the arguments of [10] hold in our case, it is sufficient to verify that the energy functional

$$F(u) = \frac{1}{2} a(u, u) + J(u) , \quad u \in V \tag{1}$$

is bounded from below on S_r, where $J : L^p(\Omega; \mathbb{R}^N) \to \mathbb{R}$ is defined by $J(u) = \int_\Omega j(x, u(x)) \, dx$. Indeed, using Lebourg's mean value theorem for locally Lipschitz functions (see [4], p. 41) we obtain

$$\begin{aligned}|j(x, y)| &\leqslant |j(x, 0)| + |j(x, y) - j(x, 0)| \\ &\leqslant |j(x, 0)| + \sup\{|w|; w \in \partial_y j(x, Y), Y \in [0, y]\} \cdot |y| \\ &\leqslant |j(x, 0)| + a_1(x)|y| + b|y|^p .\end{aligned} \tag{2}$$

Therefore

$$|J(u)| \leqslant \|j(\cdot, 0)\|_{L^1} + \|a_1\|_{L^{p'}}\|u\|_{L^p} + b\|u\|_{L^p}^p . \tag{3}$$

The continuity of the imbedding $V \subset L^p(\Omega; \mathbb{R}^N)$ ensures the existence of a positive constant $C_p(\Omega)$ such that

$$\|u\|_{L^p} \leqslant C_p(\Omega)\|u\| \quad \text{for all } u \in V .$$

From (1) and (3) it follows that

$$F_{|S_r}(u) \geqslant -\frac{1}{2}\|A\|r^2 - \|j(\cdot, 0)\|_{L^1} - C_p(\Omega)r\|a_1\|_{L^{p'}} - bC_p^p(\Omega)r^p .$$

From now on the proof follows in the same way as in [10].

Let us now consider the following non-symmetric perturbed hemivariational inequality:

$$(\mathbf{P}_2)\begin{cases}(u, \lambda) \in V \times \mathbb{R} \\ a(u, v) + \displaystyle\int_\Omega (j^0(x, u(x); v(x)) + g^0(x, u(x); v(x)))\,\mathrm{d}x \\ \quad + \langle \varphi, v\rangle_V \geqslant \lambda(u, v), \qquad \forall v \in V \\ \|u\| = r ,\end{cases}$$

where $\varphi \in V^*$ and $g : \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function which is locally Lipschitz with respect to the second variable and such that $g(\cdot, 0) \in L^1(\Omega)$. We do not make any symmetry assumption on g, but we require only the natural growth condition

($\mathbf{A}_4$) $|z| \leqslant a_2(x) + c|y|^{p-1}$,
for a.e. $(x, y) \in \Omega \times \mathbb{R}^N$ and for all $z \in \partial_y g(x, y)$,

where $a_2 \in L^{p/(p-1)}(\Omega)$ and $c > 0$.

The corresponding variant of the compactness condition ($\mathbf{A}_3$) is

($\mathbf{A}_5$) For every sequence $(u_n) \subset V$ with $\|u_n\| = r$, for every number $\alpha \in [-r^2\|A\|, r^2\|A\|]$,
and for every measurable map $z, w : \Omega \to (\mathbb{R}^N)^*$
such that $u_n \to u$ strongly in $L^p(\Omega; \mathbb{R}^N)$ for some $u \in V$,
$w(x) \in \partial_y j(x, u(x))$, $z(x) \in \partial_y g(x, u(x))$
for a.e. $x \in \Omega$ and $a(u_n, u_n) \to \alpha$

then

$$\inf_{\|\tau\|=1} \{a(\tau, \tau)\} - r^{-2}\left(\alpha + \langle \varphi, u\rangle_V + \int_\Omega \langle w(x) + z(x), u(x)\rangle \,\mathrm{d}x\right) > 0 . \tag{4}$$

Our main result asserts that the number of solutions of ($\mathbf{P}_2$) goes to infinity as the perturbation becomes smaller and smaller.

THEOREM 1. *Suppose that the assumptions* $(\mathbf{A}_1)$–$(\mathbf{A}_5)$ *hold. Then, for every* $n \geqslant 1$, *there exists* $\delta_n > 0$ *such that the problem* $(\mathbf{P}_2)$ *admits at least n distinct solutions, provided that* $\|g(\cdot, 0)\|_{L^1} \leqslant \delta_n$, $\|a_2\|_{L^{p'}} \leqslant \delta_n$, $c \leqslant \delta_n$ *and* $\|\varphi\|_{V^*} \leqslant \delta_n$.

2. Auxiliary results

We define the energy functional $H: V \to \mathbb{R}$ associated to the hemivariational problem $(\mathbf{P}_2)$ by

$$H(u) = \frac{1}{2}\, a(u, u) + J(u) + G(u) + \langle \varphi, u \rangle_V ,$$

where $G(u) = \int_\Omega g(x, u(x))\, \mathrm{d}x$. We first prove that H may be viewed as a small perturbation of the functional F and S_r.

LEMMA 1. *For every* $\varepsilon > 0$, *there exists* $\delta_\varepsilon > 0$ *such that*

$$\sup_{u \in S_r} |F(u) - H(u)| < \varepsilon ,$$

provided that $\|g(\cdot, 0)\|_{L^1} \leqslant \delta_\varepsilon$, $\|a_2\|_{L^{p'}} \leqslant \delta_\varepsilon$, $c \leqslant \delta_\varepsilon$ *and* $\|\varphi\|_{V^*} \leqslant \delta_\varepsilon$.

Proof. Proceeding in the same manner as we did for proving (2) we obtain

$$|g(x, y)| \leqslant |g(x, 0)| + a_2(x)|y| + c|y|^p .$$

Thus, for all $u \in S_r$ we have

$$\begin{aligned}|F(u) - H(u)| &\leqslant |G(u)| + |\langle \varphi, u \rangle_V| \leqslant |G(u)| + \|\varphi\|_{V^*} r \\ &\leqslant \|g(\cdot, 0)\|_{L^1} + \|a_2\|_{L^{p'}} C_p(\Omega) r + c C_p^p(\Omega) r^p + \|\varphi\|_{V^*} r < \varepsilon\end{aligned}$$

for small g, a_2, c and φ. □

Our next result shows that $H_{|S_r}$ satisfies the Palais–Smale condition in the sense of Chang [3].

LEMMA 2. *The functional H satisfies the Palais–Smale condition on* S_r.

Proof. Let (u_n) be a sequence in S_r such that

$$\sup_n |H(u_n)| < \infty \tag{5}$$

and

$$\lambda_{H_{|S_r}}(u_n) \to 0 \quad \text{as } n \to \infty , \tag{6}$$

where $\lambda_{H_{|S_r}}(u) = \min\{\|\theta\|; \theta \in \partial(H_{|S_r})(u)\}$. This functional is well defined and it is lower semicontinuous (see Clarke [4]). The expression of the generalized gradient of H on S_r is given by (see Chang [3])

$$\partial(H_{|S_r})(u) = \{\xi - r^{-2}\langle \xi, u\rangle_V \Lambda u;\quad \xi \in \partial H(u)\}\,. \tag{7}$$

From (6) and (7) we deduce the existence of a sequence $(\xi_n) \subset V^*$ such that

$$\xi_n \in \partial H(u_n) \tag{8}$$

and

$$\xi_n - r^{-2}\langle \xi_n, u_n\rangle_V \Lambda u_n \to 0, \quad \text{strongly in } V^*\,. \tag{9}$$

For every $u \in V$, the generalized gradient $\partial H(u) \subset V^*$ is given by

$$\partial H(u) = \Lambda A u + \partial(J_{|V})(u) + \partial(G_{|V})(u) + \varphi\,. \tag{10}$$

From (8), (9) and (10) it follows that there exist

$$w_n \in \partial(J_{|V})(u_n) \quad \text{and} \quad z_n \in \partial(G_{|V})(u_n)$$

such that

$$\Lambda A u_n + w_n + z_n + \varphi - r^{-2}\langle \Lambda A u_n + w_n + z_n + \varphi, u_n\rangle_V \Lambda u_n \to 0 \ \text{ strongly in } V^* \tag{11}$$

The density of V in $L^p(\Omega; \mathbb{R}^n)$ implies (see [3], Theorem 2.2)

$$\partial(J_{|V})(u) \subset \partial J(u) \quad \text{and} \quad \partial(G_{|V})(u) \subset \partial G(u)\,. \tag{12}$$

Hence

$$w_n \in \partial J(u_n) \quad \text{and} \quad z_n \in \partial G(u_n)\,. \tag{13}$$

Since V is a reflexive space and $\|u_n\| = r$ we can extract a subsequence, denoted again by (u_n) such that

$$u_n \rightharpoonup u \quad \text{weakly in } V \text{ as } n \to \infty\,. \tag{14}$$

The compactness of the imbedding $V \subset L^p(\Omega; \mathbb{R}^N)$ implies that, up to a subsequence,

$$u_n \to u \quad \text{strongly in } L^p(\Omega{:}\ \mathbb{R}^N) \text{ as } n \to \infty\,. \tag{15}$$

Using (13), (15) and the fact that the functionals J and G are locally Lipschitz on $L^p(\Omega; \mathbb{R}^N)$ we deduce the boundedness of the sequences (w_n) and (z_n) in $L^{p'}(\Omega; \mathbb{R}^N)$. Thus, passing eventually to subsequences, we have

$$w_n \rightharpoonup w \quad \text{weakly in } L^{p'}(\Omega; \mathbb{R}^N) \text{ as } n \to \infty\,. \tag{16}$$

$$z_n \rightharpoonup z \quad \text{weakly in } L^{p'}(\Omega; \mathbb{R}^N) \text{ as } n \to \infty\,. \tag{17}$$

Since the imbedding $L^{p'}(\Omega; \mathbb{R}^N) \subset V^*$ is compact relations (16) and (17) imply (up to subsequences)

$$w_n \to w \quad \text{strongly in } V^* \text{ as } n \to \infty\,. \tag{18}$$

$$z_n \to z \quad \text{strongly in } V^* \text{ as } n \to \infty\,. \tag{19}$$

Combining (14), (18) and (19) we obtain that

$$\langle w_n + z_n, u_n\rangle_V \to \langle w + z, u\rangle_V \quad \text{as } n \to \infty . \tag{20}$$

From the boundedness of u_n in V and the continuity of the bilinear form a we can suppose that, along a subsequence, we have

$$a(u_n, u_n) \to a \quad \text{as } n \to \infty \text{ for some } \alpha \in [-r^2\|A\|, r^2\|A\|] .$$

Taking into accound (18), (19) and (20) we see that (11) implies

$$Au_n - r^{-2}(\alpha + \langle \varphi, u\rangle_V + \langle w + z, u\rangle_V)u_n \text{ converges in } V \text{ as } n \to \infty . \tag{21}$$

Using (13) and (15)–(17) and the fact that the Clarke generalized gradient is a weak*-closed multifunction (see [4], Proposition 2.1.5) we deduce

$$w \in \partial J(u) \tag{22}$$

$$z \in \partial G(u) . \tag{23}$$

Our hypotheses ($\mathbf{A}_2$) and ($\mathbf{A}_4$) allow to apply Theorem 2.7.5 in [4] and from relations (22) and (23) we get the existence of two measurable mappings $w, z : \Omega \to (\mathbb{R}^N)^*$ such that

$$w(x) \in \partial_y j(x, u(x)) \quad \text{for a.e. } x \in \Omega \tag{24}$$

$$z(x) \in \partial_y g(x, u(x)) \quad \text{for a.e. } x \in \Omega \tag{25}$$

$$\langle w, u\rangle_V = \langle w, u\rangle_{L^p(\Omega;\mathbb{R}^N)} = \int_\Omega \langle w(x), u(x)\rangle \, dx . \tag{26}$$

$$\langle z, u\rangle_V = \langle z, u\rangle_{L^p(\Omega;\mathbb{R}^N)} = \int_\Omega \langle z(X), u(x)\rangle \, dx \tag{27}$$

Remark that due to ($\mathbf{A}_2$), (24) and $u \in L^p(\Omega; \mathbb{R}^N)$ we have that $\langle w(x), u(x)\rangle \in L^1(\Omega; \mathbb{R})$. Indeed,

$$\int_\Omega |\langle w(x), u(x)\rangle| \, dx \leqslant \int_\Omega (a_1(x) + b|u(x)|^{p-1})|u(x)| \, dx \leqslant \|a_1\|_{L^{p'}} \cdot \|u\|_{L^p} + b\|u\|^p_{L^p} .$$

In the same way using ($\mathbf{A}_4$), (25) and $u \in L^p(\Omega; \mathbb{R}^N)$ we obtain that $\langle z(x), u(x)\rangle \in L^1(\Omega; \mathbb{R})$. Replacing (26) and (27) in (21) one gets that

$$Au_n - r^{-2}(\alpha + \langle \varphi, u\rangle_V + \int_\Omega \langle w(x) + z(x), u(x)\rangle \, dx)u_n \text{ converges in } V \text{ as } n \to \infty \tag{28}$$

with w (resp., z) satisfying (24) (resp., (25)). Consequently, we are in the position to use assumption ($\mathbf{A}_5$) and therefore inequality (4) is valid. For all n, k we have

$$\left(\inf_{\|\tau\|=1}\{a(\tau,\tau)\} - r^{-2}\left(\alpha + \langle\varphi, u\rangle_V + \int_\Omega \langle (w+z)(x), u(x)\rangle \, dx\right)\right)$$
$$\times \|u_n - u_k\|^2 \leqslant a(u_n - u_k, u_n - u_k) - r^{-2}$$
$$\times\left(\left(\alpha + \langle\varphi, u\rangle_V + \int_\Omega \langle (w+z)(x), u(x)\rangle \, dx\right)(u_n - u_k), u_n - u_k\right)$$
$$= \left(A(u_n - u_k) - r^{-2}\left(\alpha + \langle\varphi, u\rangle_V + \int_\Omega \langle (w+z)(x), u(x)\rangle \, dx\right)\right.$$
$$\left.(u_n - u_k), u_n - u_k\right) \leqslant \|A(u_n - u_k) - r^{-2}$$
$$\left(\alpha + \langle\varphi, u\rangle_V + \int_\Omega \langle (w+z)(x), u(x)\rangle \, dx\right)(u_n - u_k)\| \cdot \|u_n - u_k\| \, .$$

The convergence in (28), the above estimates and (4) show that (u_n) contains a Cauchy subsequence in V. Hence u_n converges along a subsequence in V to u. This completes the proof of the lemma. □

The next result shows that H plays indeed the role of energy functional for the perturbed problem $(\mathbf{P}_2)$.

LEMMA 3. *If u is a critical point of $H_{|S_r}$ then there exists $\lambda \in \mathbb{R}$ such that (u, λ) is a solution of problem $(\mathbf{P}_2)$.*

Proof. Since $0 \in \partial(H_{|S_r})(u)$ it follows by (7), (10) and (12) that there exist

$$w \in \partial(J_{|V})(u) \subset \partial J(u) \quad \text{and} \quad z \in \partial(G_{|V})(u) \subset \partial G(u) \tag{29}$$

such that u is a solution of

$$\Lambda A u + w + z + \varphi = r^{-2}\langle \Lambda A u + w + z + \varphi, u\rangle_V \Lambda u \, . \tag{30}$$

From Theorem 2.7.3 in [4] we have that for every $u \in L^p(\Omega; \mathbb{R}^N)$,

$$\partial J(u) \subset \int_\Omega \partial_y j(x, u(x)) \, dx \quad \text{and} \quad \partial G(u) \subset \int_\Omega \partial_y g(x, u(x)) \, dx.$$

Thus, by (29), the mappings $w, z : \Omega \to (\mathbb{R}^N)^*$ satisfy

$$w(x) \in \partial_y j(x, u(x)) \quad \text{for a.e. } x \in \Omega \, , \tag{31}$$

$$z(x) \in \partial_y g(x, u(x)) \quad \text{for a.e. } x \in \Omega \, , \tag{32}$$

and, for all $v \in V$,

$$\langle w, v\rangle_V = \int_\Omega \langle w(x), v(x)\rangle \, dx \, , \tag{33}$$

$$\langle z, v\rangle_V = \int_\Omega \langle z(x), v(x)\rangle \, dx \, . \tag{34}$$

Let us take

$$\lambda = r^{-2}\left(\langle \Lambda Au + \varphi, u\rangle_V + \int_\Omega \langle w(x) + z(x), u(x)\rangle \, \mathrm{d}x\right). \tag{35}$$

From (30)–(35) it follows that, for every $v \in V$

$$\begin{aligned}
\lambda(u, v) - a(u, v) - \langle \varphi, v\rangle_V &= \int_\Omega \langle w(x) + z(x), v(x)\rangle \, \mathrm{d}x \\
&\leqslant \int_\Omega \max\{\langle \mu_1(x), v(x)\rangle;\ \mu_1(x) \in \partial_y j(x, u(x))\} \, \mathrm{d}x \\
&\quad + \int_\Omega \max\{\langle \mu_2(x), v(x)\rangle;\ \mu_2(x) \in \partial_y g(x, u(x))\} \, \mathrm{d}x \\
&= \int_\Omega j^0(x, u(x); v(x)) \, \mathrm{d}x + \int_\Omega g^0(x, u(x); v(x)) \, \mathrm{d}x .
\end{aligned}$$

In order to write the last equality we have used Proposition 2.1.2 from [4]. The proof of this lemma is now complete. □

3. Trivial pairs and essential values

In what follows, X denotes a metric space, A is a subset of X and i stands for the inclusion map of A in X. For the topological notions mentioned in this section we refer to [5, 6, 20].

DEFINITION 1. A map $r : X \to A$ is said to be r retraction if it is continuous, surjective and $r_{|A} = Id$.

DEFINITION 2. A retraction r is called a strong deformation retraction provided that there exists a homotopy $\zeta : X \times [0, 1] \to X$ of $i \circ r$ and $\mathbf{1}_X$ which satisfies the additional condition $\zeta(x, t) = \zeta(x, 0)$, for any $(x, t) \in A \times [0, 1]$.

DEFINITION 3. The metric space X is said to be weakly locally contractible, if for every $u \in X$ there exists a neighbourhood U of u contractible in X.

For every $a \in \mathbb{R}$, denote

$$f^a = \{u \in X : f(u) \leqslant a\} ,$$

where $f : X \to \mathbb{R}$ is a continuous function.

DEFINITION 4. Let $a, b \in \mathbb{R}$ with $a \leqslant b$. The pair (f^b, f^a) is said to be trivial provided that, for every neighbourhood $[a', a'']$ of a and $[b', b'']$ of b, there exist some closed sets A and B such that $f^{a'} \subseteq A \subseteq f^{a''}$, $f^{b'} \subseteq B \subseteq f^{b''}$ and such that A is a strong deformation retraction of B.

DEFINITION 5. (Degiovanni-Lancelotti [6]). A real number c is an essential value of f provided that, for every $\varepsilon > 0$ there exist $a, b \in (c-\varepsilon, c+\varepsilon)$ with $a < b$ such that the pair (f^b, f^a) is not trivial.

The following property of essential values is due to Degiovanni-Lancelotti (see [6], Theorem 2.6).

PROPOSITION 1. *Let c be an essential value of f. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that every continuous function $g : X \mapsto \mathbb{R}$ with*

$$\sup\{|g(u) - f(u)| : u \in X\} < \delta$$

admits an essential value in $(c-\varepsilon, c+\varepsilon)$.

For every $n \geqslant 1$, define

$$\Gamma_n = \{S \subset S_r;\, S \subset \mathscr{F},\, \gamma(S) \geqslant n\},$$

where $\mathscr{F}$ is the class of closed symmetric subsets of S_r with respect to the origin and $\gamma(S)$ represents the Krasnoselskii genus of $S \in \Gamma_n$, i.e. the smallest $k \in \mathbb{N} \cup \{+\infty\}$ for which there exists a continuous and odd map from S into $\mathbb{R}^k \setminus \{0\}$. Motreanu and Panagiotopoulos proved in [10] that the corresponding min-max values of F over Γ_n

$$\beta_n = \inf_{S \in \Gamma_n} \max_{u \in S} F(u), \quad n \geqslant 1,$$

are critical values of F on S_r.

PROPOSITION 2. *We have that* $\sup_{S_r} F$ *is not achieved and* $\lim_{n\to\infty} \beta_n = \sup_{u\in S_r} F(u)$. *Moreover, there exists a sequence (b_n) of essential values of $F_{|S_r}$ strictly increasing to* $\sup_{u\in S_r} F(u)$.

The proof of this result is essentially contained in Degiovanni-Lancelotti [6].

4. Proof of Theorem 1

Let $n \geqslant 1$ be fixed. From Lemma 3 we see that it is sufficient to prove the existence of some $\delta_n > 0$ such that the functional $H_{|S_r}$ has at least n distinct critical values, provided that $\|g(\cdot, 0)\|_{L^1} \leqslant \delta_n, \|a_2\|_{L^{p'}} \leqslant \delta_n$, $c \leqslant \delta_n$ and $\|\varphi\|_{V^*} \leqslant \delta_n$. In view of Proposition 2, we can find a sequence (b_n) of essential values of $F_{|S_r}$ which increases strictly to $\sup_{u\in S_r} F(u)$. Let $\varepsilon_0 > 0$ be chosen such that $\varepsilon_0 < 1/2 \min_{2\leqslant i\leqslant n} (b_i - b_{i-1})$. We apply Proposition 1 to $F_{|S_r}$ and $H_{|S_r}$. Hence, for every $1 \leqslant j \leqslant n$, there exists $\eta_j > 0$ such that

$$\sup_{u\in S_r} |F(u) - H(u)| < \eta_j$$

implies the existence of an essential value c_j of $H_{|S_r}$ in $(b_j - \varepsilon_0, b_j + \varepsilon_0)$. Applying Lemma 1 for $\eta = \min\{\eta_1, \dots, \eta_n\}$ we get the existence of some $\delta_n > 0$ such that

$$\sup_{u \in S_r} |F(u) - H(u)| < \eta ,$$

provided that $\|g(\cdot, 0)\|_{L^1} \leq \delta_n$, $\|a_2\|_{L^{p'}} \leq \delta_n$, $c \leq \delta_n$ and $\|\varphi\|_{V^*} \leq \delta_n$. Therefore the functional $H_{|S_r}$ has at least n distinct essential values $c_1, c_2, \dots, c_n$ in $(b_1 - \varepsilon_0, b_n + \varepsilon_0)$. We now show that $c_1, c_2, \dots, c_n$ are critical values of $H_{|S_r}$. Assuming the contrary, there exists some $j \in \{1, 2, \dots, n\}$ such that c_j is not a critical value of $H_{|S_r}$. In what follows we are going to prove that

(A) There exists $\bar{\varepsilon} > 0$ so that $H_{|S_r}$ has no critical value in $(c_j - \bar{\varepsilon}, c_j + \bar{\varepsilon})$;

(B) For every $a, b \in (c_j - \bar{\varepsilon}, c_j + \bar{\varepsilon})$ with $a < b$, the pair $(H^b_{|S_r}, H^a_{|S_r})$ is trivial.

Suppose that (A) is not valid. Then we get the existence of a sequence (d_k) of critical values of $H_{|S_r}$ with $d_k \to c_j$ as $k \to \infty$. Since d_k is a critical value it follows that there exists $u_k \in S_r$ such that

$$H(u_k) = d_k \quad \text{and} \quad \lambda_{H|S_r}(u_k) = 0 .$$

Using the fact that $(PS)_{c_j}$ holds we can suppose that, up to a subsequence, (u_k) converges to some $u \in S_r$ as $k \to \infty$. Taking into account the continuity of H and the lower semi-continuity of $\lambda_{H_{|S_r}}$ we obtain

$$H(u) = c_j \quad \text{and} \quad \lambda_{H_{|S_r}}(u) = 0 ,$$

which contradicts the initial assumption on c_j.

To get (B) we apply the Noncritical Point Theorem (see [5], Theorem 2.15) which implies that there exists a continuous map $\chi : S_r \times [0, 1] \mapsto S_r$ such that

$$\begin{aligned} &\chi(u, 0) = u, \qquad H(\chi(u, t)) \leq H(u) , \\ &H(u) \leq b \Rightarrow H(\chi(u, 1)) \leq a , \qquad H(u) \leq a \Rightarrow \chi(u, t) = u . \end{aligned} \tag{36}$$

Define the map ρ as folows:

$$\rho : H^b_{|S_r} \mapsto H^a_{|S_r} , \qquad \rho(u) = \chi(u, 1) .$$

From (36) we have that ρ is well defined and it is a retraction. Set

$$\mathscr{H} : H^b_{|S_r} \times [0, 1] \mapsto H^b_{|S_r} , \qquad \mathscr{H}(u, t) = \chi(u, t) .$$

We easily see that, for every $u \in H^b_{|S_r}$

$$\mathscr{H}(u, 0) = u \quad \text{and} \quad \mathscr{H}(u, 1) = \rho(u) \tag{37}$$

and, for each $(u, t) \in H^a_{|S_r} \times [0, 1]$,

$$\mathscr{H}(u, t) = \mathscr{H}(u, 0) . \tag{38}$$

From (37) and (38) it follows that $\mathscr{H}$ is a $H^a_{|S_r}$-homotopic to the identity of $H^a_{|S_r}$, i.e. $\mathscr{H}$ is a strong deformation retraction, hence the pair $(H^b_{|S_r}, H^a_{|S_r})$ is trivial. Assertions (A), (B) and Definition 5 show that c_j is not an essential value of $H_{|S_r}$ which concludes our proof. □

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